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Exact tachyon condensation on non-commutative torus

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Abstract: We construct the exact non-commutative solutions on tori. This gives an exact description of tachyon condensation on bosonic D-branes, non-BPS D-branes and brane-antibrane systems. We obtain various bound states of D-branes after the tachyon condensation. Our results show that these solutions can be generated by applying the gauge Morita equivalence between the constant curvature projective modules. We argue that there is a general framework of the non-commutative geometry based on the notion of Morita equivalence which underlies this specific example.

Keywords: Superstrings and Heterotic Strings, Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

Contents

1. Introduction

D-branes have been playing the most prominent rôle in the recent developments of string theory. This is not only because they are the non-perturbative objects but also because they possess many interesting characteristics which require mathematically distinguished descriptions. It is very interesting to try to prove that these various viewpoints are mutually consistent. Such an effort often gives profound relationships between physics and mathematics.

One of such examples is the quantum field theory on the non-commutative space arising from open string theory in B-field background [1, 2]. In the commutative case, the space of functions gives the basis of the quantum field theory, for example, through the mode expansion. In the non-commutative space they are replaced by the non-commutative C^*-algebra $\mathcal{A}$ which is at the heart of the principle of the non-commutative geometry (see for example [3]). In this paper we are mainly interested in the description of D-branes with the direct use of such a principle.

In the commutative approach, the soliton charges of D-branes are derived from the topological K-group [4, 5]. It is based on the observation that there are always massless gauge particles which define the vector bundle on D-branes. We need to take the formal difference of vector bundles to describe the brane-antibrane pair annihilation process [6, 7].

In non-commutative geometry, the topological information of the "manifold" is given by the operator algebra version of the K-group $K(\mathcal{A})$. For example in Connes' index theorem [3], the topological index is give by the pairing of the element of K-group with that of the cyclic cohomology group of $\mathcal{A}$. We note that $K_0(\mathcal{A})$ is defined by an equivalence class of projection operators in $\mathrm{Mat}_\infty(\mathcal{A})$, the infinite-dimensional matrix algebra with elements in $\mathcal{A}$.

From physical viewpoints, it is natural to conjecture that the topological properties of D-branes are described by the operator algebra K-group. It is described by the solitonic configurations which are proportional to projection operators. Remarkably such configurations indeed appear as classical solutions (GMS soliton) in the scalar field theory on the non-commutative plane (Moyal plane) in the large noncommutativity limit [8]. This idea was immediately applied to string theory in the tachyon condensation process [6, 7], e.g. bosonic D-branes, non-BPS D-branes and brane-antibrane systems [9, 10]. If we use non-trivial GMS soliton, the lower-dimensional D-branes are generated. As pointed out in [11, 12] it can be seen as the non-commutative generalization [13] of the correspondence between D-brane charges and K-theory [4, 5].

In this way the unstable system of D-branes seems to give a good example for the application of the geometrical methods of the non-commutative geometry to the string theory. There are two directions to proceed. One is to challenge non-commutative spaces with more complicated and richer structures. The simplest non-trivial example is the non-commutative torus. This example is interesting from physical side since we expect to have analogue of T-duality symmetry in the form of Morita equivalence. It was investigated in [14, 15] by using Powers-Rieffel projection. Unlike Moyal plane, we observed a sort of instability [14]. It comes from the fact that we may construct the non-commutative soliton with arbitrary small size. Mathematically it is related to the fact that K-group of non-commutative tori is not quantized in $\mathbb{Z}$ but takes its value in $\mathbb{R}$. It remained as a puzzle whether it is natural to interpret the continuous value as the D-brane charge. Later a different construction of the soliton configuration on the torus and on the orbifold was discussed in [16]. Among other things, a remarkable suggestion is to use Morita equivalence bimodule in the construction of the non-commutative soliton. Similar construction of the projection operator on the tori was also discussed in [17, 18]. For fuzzy sphere, GMS-like solitons were discussed in [19].

The other direction is to take the gauge field into account and to construct the exact solution without taking the large-B limit. It was pioneered in [20] when the base space is Moyal plane. Certain constraint on the coupling of field strength and tachyon field should be satisfied in order to have such property.

In the present paper we continue to study the non-commutative soliton on the two-torus. We have mentioned two motivations, (1) how to resolve instability of the spectrum and (2) the construction of exact solution. The use of Morita equivalence initiated in [16] gives another motivation. For the non-trivial examples such as tori, we can not directly construct the analogue of the shift operator. In this sense, we can not escape from using more abstract Morita equivalence bimodule directly to construct the non-commutative soliton. (A nice review of Morita equivalence for non-commutative torus is given by [21]). Once we know how to use it, one may apply the method to other examples as well, namely in the generic examples of the open string systems interpolating D-branes [2, 13]. We argue that the Morita equivalence gives a natural generalization of the notion of the brane-antibrane systems and leads to the description which is similar to the superconnection [5, 22, 23, 24]. Inspired by this fact we propose an equation which defines the non-commutative solitons on brane-antibrane systems in the generalized sense.

As we will see, we can obtain the exact solutions of the tachyon condensation on the two-torus by employing the constant curvature connections. In the non-commutative torus, the constant curvature connection parameterizes the equivalence class of the whole projective modules (analogue of vector bundle). As a result we obtain various D2-D0 bound states after the tachyon condensation on a non-BPS D2-brane. We also find that the gauge Morita equivalence [21, 25] (bimodule between the constant curvature connections) plays a crucial rôle in generating solutions. In fact one can construct exact solutions for any C^*-algebras if there exist the gauge Morita equivalence bimodules. Furthermore this exact analysis explicitly shows that for finite B/g the above mentioned instability does not occur.

The paper is organized as follows. In section 2, we discuss tachyon condensation on generic non-commutative spaces. After we review the Morita equivalence, we construct the projection operator of the brane-antibrane systems in the generalized sense. We see that the structure of superconnection [26] naturally appears as the linking algebra in the framework of C^*-algebra. In section 3 we discuss the tachyon condensation on the non-commutative two-torus. We construct exact solutions for bosonic D-branes, non-BPS branes and brane-antibrane systems in terms of the flat curvature connections. We also discuss the solution generating rule for this examples by using the gauge Morita equivalence. This section also includes a review of some mathematical results on the projective modules. In section 4 we summarize the conclusions. In appendix A we give a review of the explicit example of projections in non-commutative tori and we also show the calculations of their topological charges.

2. Morita equivalence and non-commutative soliton: a general strategy

We start from discussing relatively formal viewpoint which will be useful in the later sections. While our main result is restricted to the non-commutative solitons on non-commutative tori in section 3, we think that our method can be basically applied as well to the other open string systems such as $Dp - Dp'$.

Let us first recall the definition of the non-commutative solitons [8]. They are the solutions to the equation of motion,

$$\frac{\partial V(\star\phi)}{\partial\phi} = 0\,. \tag{2.1}$$

Here $\star$ is the (non-commutative) product of given C^*-algebra $\mathcal{A}^{(0)}$ which defines the non-commutative geometry on the single D-brane. It is solved in the following form,

$$\phi(x) = \sum_i \lambda_i P_i\,, \qquad \lambda_i \in \mathbb{R}\,, \qquad P_i \in \mathcal{A}^{(0)}\,, \tag{2.2}$$

where λ_is are the solutions to the equation $\partial V(\lambda)/\partial\lambda = 0$ and P_i's are the mutually orthogonal projections $P_i \cdot P_j = \delta_{ij} P_i$, $P_i^* = P_i$. In this sense, the construction of non-commutative solitons is reduced to finding the projection operators.[1]

In the mathematical context, the classification of the projection operator is directly related to the definition of the K_0-group of the operator algebra K-theory. This is related to the fact that the non-commutative analogue of the vector bundle is described by the projective module of given C^*-algebra $\mathcal{A}$. Let us briefly illustrate the correspondence by using the relationship with the commutative theory.

Let M be a smooth manifold (base space) and E be a vector bundle over M. In the context of the topological K-theory, it is known that the isomorphism class of E is an element of K^0-group.[2] By Swan's theorem, for any E there exists a vector bundle F such that $E \oplus F$ is a trivial bundle over M. Let $C^\infty(M)$ be the smooth function on M. The trivial bundle can be written as $(C^\infty(M))^N$ with some $N \in \mathbb{N}$. Therefore, any vector bundle over M can be obtained by acting P on $(C^\infty(M))^N$, where P is a projection in $\mathrm{Mat}_N(C^\infty(M))$.[3] The module which is constructed by applying the projection operator to the free module is called projective module.

This characterization of the vector bundle can be generalized to non-commutative theory. A non-commutative algebra $\mathcal{A}^{(0)}$ replaces $C^\infty(M)$ for "non-commutative space". The free module $(\mathcal{A}^{(0)})^N$ corresponds to the rank N trivial bundle on commutative space. Projective module E is defined as $\mathcal{A}^{(0)}$-module such that there exists the other $\mathcal{A}^{(0)}$-module F with $E \oplus F = (\mathcal{A}^{(0)})^N$. Thus the non-commutative analogue of a

[1] If the number of the D-branes is greater than one (say N), we need to consider the projector in the matrix algebra $\mathrm{Mat}_N(\mathcal{A})$.

[2] In general, the formal difference of the isomorphism class of vector bundles over M is the element of K^0.

[3] We define the algebra $\mathrm{Mat}_N(\mathcal{A})$ as the N times N matrix algebra with elements in $\mathcal{A}$.

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vector bundle is the projective module $E = P(\mathcal{A}^{(0)})^N$ which is defined by a projection operator $P \in \mathrm{Mat}_N(\mathcal{A}^{(0)})$. In this sense, D-branes on non-commutative spaces are described by the projective modules, and the operator algebra K_0-group [11, 12, 13].

Let us come back to the issue of the construction of the projection operator. We would like to use the Morita equivalence bimodule as the abstract building block to construct non-commutative soliton.

Morita equivalence is one of the central idea of the classification of C^*-algebra. From the mathematical viewpoint, it is essential to determine when two C^*-algebras $\mathcal{A}, \mathcal{B}$ define the same type of non-commutative geometry. In the non-commutative geometry, the idea of points is replaced by the set of ideals of C^*-algebra. It is then known that two C^*-algebras $\mathcal{A}, \mathcal{B}$ have the same set of ideals if there are $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$ (for example see [27]). It is defined as a bimodule on which $\mathcal{A}$ (resp. $\mathcal{B}$) acts from the left (resp. right). It also needs to have two types of inner products $\langle\ ,\ \rangle_{\mathcal{A}}$, $\langle\ ,\ \rangle_{\mathcal{B}}$ with value in $\mathcal{A}$ and $\mathcal{B}$, respectively with the following conditions

$$\begin{aligned}
&\langle ax, y\rangle_{\mathcal{A}} = a\langle x, y\rangle_{\mathcal{A}}\,, \qquad \langle x, y\rangle^*_{\mathcal{A}} = \langle y, x\rangle_{\mathcal{A}}\,, \qquad a \in \mathcal{A}\,,\\
&\langle x, yb\rangle_{\mathcal{B}} = \langle x, y\rangle_{\mathcal{B}} b\,, \qquad \langle x, y\rangle^*_{\mathcal{B}} = \langle y, x\rangle_{\mathcal{B}}\,, \qquad b \in \mathcal{B}\,,\\
&\langle xb, y\rangle_{\mathcal{A}} = \langle x, yb^*\rangle_{\mathcal{A}}\,, \qquad \langle ax, y\rangle_{\mathcal{B}} = \langle x, a^* y\rangle_{\mathcal{B}}\,.
\end{aligned} \tag{2.3}$$

The most important property which should be satisfied by them is the associativity

$$\langle x, y\rangle_{\mathcal{A}} z = x\langle y, z\rangle_{\mathcal{B}}\,, \qquad x, y, z \in {}_{\mathcal{A}}X_{\mathcal{B}}\,. \tag{2.4}$$

Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ which are related by the bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$ are called Morita equivalent.

In string theory, there is a natural interpretation of such equivalence relation. It is well known that the noncommutativity arises on D-branes connected by the open string in the presence of B-field. On the two ends of the open string, we have two D-branes and generally two different types of non-commutative geometry. Suppose they are defined by C^*-algebras $\mathcal{A}, \mathcal{B}$.[4] Under such circumstances, it is natural to conjecture (for example, see [2, 13]),

- The bimodule naturally interpreted as the open string field Ψ where $\mathcal{A}$ (resp. $\mathcal{B}$) acts from the left (resp. right).
- Two inner products are identified the product of open string fields. We have two because we have two choices (which side of the open string) for the contraction.
- The associativity of the product corresponds to that of the open strings.

[4]More precisely, let $\mathcal{A}^{(0)}$ be the algebra corresponding to the non-commutative base space, and let E_α (resp. E_β) be the projective $\mathcal{A}^{(0)}$-module (D-brane) related to $\mathcal{A}$ (resp. $\mathcal{B}$), then $\mathcal{A} = \mathrm{End}_{\mathcal{A}^{(0)}} E_\alpha$ and $\mathcal{B} = \mathrm{End}_{\mathcal{A}^{(0)}} E_\beta$. Let $a \in \mathcal{A}^{(0)}$, $A \in \mathrm{End}_{\mathcal{A}^{(0)}} E$ and $\xi \in E$. $\mathrm{End}_{\mathcal{A}^{(0)}} E$ means $(a \cdot \xi)A = a(\xi \cdot A)$. This is the natural non-commutative generalization of the definition of endomorphisms for vector bundles.

Although the actual justification of these statements is far from being obvious at this stage, it gives a nice intuition to the otherwise abstract nature of Morita equivalence.

In the following, we use the Morita equivalence bimodule to define the projection operator (= non-commutative soliton). In an abstract language, it can be described as follows [16, 27, 28, 29]. In the very definition of the Morita equivalence, we actually need to impose that the inner product $\langle\ ,\ \rangle_{\mathcal{A},\mathcal{B}}$ maps ${}_{\mathcal{A}}X_{\mathcal{B}}$ into a dense set of the $\mathcal{A},\mathcal{B}$, respectively. Namely for any operator $a\in\mathcal{A}$, there should be a finite set $x_i, y_i\in {}_{\mathcal{A}}X_{\mathcal{B}}$ such that $a=\sum_i\langle x_i,y_i\rangle_{\mathcal{A}}$. Suppose $\mathcal{A}$ has an identity element and choose x_i, y_i such that $1=\sum_{i=1}^N\langle x_i,y_i\rangle_{\mathcal{A}}$. From x_i, y_i one may define the projection operator in $\mathrm{Mat}_N(\mathcal{B})$ as $P\equiv\langle y_i,x_j\rangle_{\mathcal{B}}$ since

$$\begin{aligned}\sum_j\langle y_i,x_j\rangle_{\mathcal{B}}\langle y_j,x_k\rangle_{\mathcal{B}} &= \sum_j\langle y_i,x_j\langle y_j,x_k\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\\ &= \left\langle y_i,\sum_j\langle x_j,y_j\rangle_{\mathcal{A}}x_k\right\rangle_{\mathcal{B}} = \langle y_i,x_k\rangle_{\mathcal{B}}\,. \end{aligned} \tag{2.5}$$

Unlike the original GMS soliton where the projection operator defines the lower-dimensional D-branes, it seems rather hard to identify the nature of the projected space. However, from the mathematical side, $\mathcal{A}$ can be embedded into $\mathrm{Mat}_N(\mathcal{B})$, as

$$\mathcal{A}\sim P\cdot\mathrm{Mat}_N(\mathcal{B})\cdot P\,. \tag{2.6}$$

Suppose $\mathcal{A}$ (resp. $\mathcal{B}$) describes the non-commutative geometry on D-brane α (resp. β). Then a physical interpretation of above identity is that α can appear as the non-commutative soliton on the N copies of the D-branes β through the tachyon condensation process in a generalized sense.

In order that this type of interpretation is possible, we need the analogue of $D\bar{D}$ pair for this generalized setting. In the study of $D\bar{D}$ pair, the tachyon condensation process is described by the combination of gauge fields and tachyon fields (so called "superconnection" [26]) as argued in [5, 22, 23, 24],

$$\begin{pmatrix} d+A_1 & T\\ \bar{T} & d+A_2\end{pmatrix}, \tag{2.7}$$

where each entry represents the various sector of the open string.

There exists an analogue of superconnection in C^*-algebra which is called the linking algebra $\mathcal{C}$ (see for example, [27]). It is defined in such way as containing the algebras $\mathcal{A},\mathcal{B}$ as its complementary components. Namely there exists a projection $P\in\mathcal{C}$ such that $\mathcal{A}=P\cdot\mathcal{C}\cdot P$ and $\mathcal{B}=(1-P)\cdot\mathcal{C}\cdot(1-P)$. It is known that the linking algebra $\mathcal{C}$ exists if and only if two C^*-algebras $\mathcal{A},\mathcal{B}$ are Morita equivalent. Namely if the bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$ exists, one may define the linking algebra by matrices,

$$\begin{pmatrix} a & x\\ \bar{y} & b\end{pmatrix},\qquad a\in\mathcal{A}\,,\qquad b\in\mathcal{B}\,,\qquad x,y\in{}_{\mathcal{A}}X_{\mathcal{B}}\,. \tag{2.8}$$

One may easily check that the matrix multiplication of such 2×2 matrices is well-defined and the obvious projection operator $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives the projection into $\mathcal{A}$. By comparing (2.7) and (2.8) one sees that the Morita equivalence bimodule plays the analogous rôle as that of the tachyon field and that the two gauge fields correspond to the endomorphism algebras $\mathcal{A} = \mathrm{End}_{\mathcal{A}^{(0)}} E_\alpha$ and $\mathcal{B} = \mathrm{End}_{\mathcal{A}^{(0)}} E_\beta$.

This analogy was implicitly used in the explicit calculation of the tachyon condensation of $D\bar{D}$ system in the Moyal plane [20, 30]. We summarize our conventions for Moyal plane in appendix A. We recall that the non-commutative soliton in this case was constructed out of the partial isometry described by the tachyon fields T and $\bar{T}$ which satisfy,

$$T\bar{T}T = T\,, \qquad \bar{T}T\bar{T} = \bar{T}\,. \tag{2.9}$$

With these relations, $T\bar{T}$ and $\bar{T}T$ become projection operators and are identified as the non-commutative soliton. Explicitly T can be constructed by using the shift operator $S = \sum_n |n\rangle\langle n+1|$, which satisfies $S\bar{S} = 1$ and $\bar{S}S = 1 - |0\rangle\langle 0|$ [20]. The tachyon field is given by $T = S^m$ ($m \in \mathbb{N}$) with this solution generating rule. In this case, T gives the isometry between the subsets of the identical algebra $\mathcal{A}$ and $\mathcal{A}$ acts on T from the both sides.

In our more general situation, the algebra acting on T from left is in general different from that acting from right. In this sense it is natural to replace the partial isometry by the Morita equivalence bimodule. The statement which corresponds to (2.9) is that one may choose an element T of the bimodule which satisfies

(1) $P = \langle T, T\rangle_{\mathcal{A}}$ and $Q = \langle T, T\rangle_{\mathcal{B}}$ are the projectors of $\mathcal{A}, \mathcal{B}$.

(2) It satisfies the analog of the partial isometry relation,

$$\langle T, T\rangle_{\mathcal{A}} T = T\langle T, T\rangle_{\mathcal{B}} = T\,. \tag{2.10}$$

While it is a non-trivial question whether we may find such T, the requirement of these two conditions are actually equivalent [28]. Suppose that the partial isometry-like equation (2.10) holds. $P = \langle T, T\rangle_{\mathcal{A}}$ and $Q = \langle T, T\rangle_{\mathcal{B}}$ are clearly self-adjoint, and direct calculation shows that $P^2 = P$ and $Q^2 = Q$ by using the properties of the two inner products (2.3) and (2.4). Conversely, if we start from the condition $P^2 = P$, the partial isometry-like equations follows from vanishing of the norm,

$$\langle \langle T, T\rangle_{\mathcal{A}} T - T,\ \langle T, T\rangle_{\mathcal{A}} T - T\rangle_{\mathcal{A}} = 0\,. \tag{2.11}$$

One can also see from this proof that $P^2 = P$ implies $Q^2 = Q$.

We would like to propose that eq. (2.10) defines the non-commutative soliton on the brane-antibrane systems in the generalized sense. We will see the explicit examples on non-commutative tori in section 3.4.

3. Non-commutative torus and D-branes

Because our discussions so far are given in an abstract language, it is desirable to investigate explicit examples in order to illuminate the idea. The simplest example is D-branes on non-commutative plane (Moyal plane) but it is too simple to use the machinery we would like to examine since it reduces to the (infinite-dimensional) matrix algebra.Thus in this section we consider D-branes on non-commutative tori, where Morita equivalence is interpreted as T-duality [1, 25, 31, 32] and has a rich structure.

The algebra of non-commutative two-torus is generated by unitary elements U_1 and U_2 with the relation,

$$U_1 U_2 = U_2 U_1 e^{2\pi i\theta}\,, \tag{3.1}$$

where the real number $\theta \in [0, 1]$ is the parameter of the algebra. When θ is rational, the algebra reduces to the matrix algebra of finite size. Therefore we assume that θ is irrational. We write this algebra by $\mathcal{A}_\theta$. The generators can be written in terms of non-commutative coordinates (x^1, x^2) with $[x^1, x^2] = -2\pi i\alpha'\theta$ as follows

$$U_1 = e^{ix^1/\sqrt{\alpha'}}\,, \qquad U_2 = e^{ix^2/\sqrt{\alpha'}}\,. \tag{3.2}$$

A generic element $a \in \mathcal{A}_\theta$ can be expanded by U_i as[5]

$$a = \sum_{m,n\in\mathbb{Z}} a_{mn} U_1^m U_2^n\,. \tag{3.3}$$

While we can not realize this algebra as the matrix algebra for the irrational θ, one may formally define a trace for $\mathcal{A}_\theta$ as follows

$$\mathrm{Tr}\, a = a_{00}\,, \tag{3.4}$$

by using the above expansion. This is equal to the integration over (x^1, x^2). It is obvious that it satisfies the fundamental relation $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

As we saw in section 2, the gauge bundle on the D-brane is described by the projective $\mathcal{A}_\theta$-module. It is known that the isomorphic class of the projective modules on a non-commutative torus is classified by Chern character [29] which specifies the element of $K_0(\mathcal{A}_\theta)$. The relation of these mathematical facts to the RR-couplings of brane-antibrane systems will be discussed later in the subsection 3.4.

In this section, after explaining some mathematical backgrounds on the projective module on the non-commutative tori, we discuss mass spectrum of BPS D-branes and their T-duality transformation. Finally we investigate tachyon condensation on non-BPS D-branes and brane-antibrane systems.

[5] To be exact we should assume a_{mn} tend to zero faster than any powers of $|n| + |m|$ as $|n| + |m| \to \infty$.

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3.1 Projective modules, constant curvature connection and morita equivalence on non-commutative torus

In this subsection, we explain some mathematical results on $\mathcal{A}_\theta$, especially the projections, the projective modules, the constant curvature connections and Morita equivalence[6] which will be essential for later arguments. The readers who are familiar with these materials may skip this subsection.

We start from defining the projection operators on $\mathcal{A}_\theta$. The unitary equivalence class[7] of projections is characterized by the value of the trace Tr [28, 33] as follows

$$\mathrm{Tr}(P_{n-m\theta}) = n - m\theta\,, \qquad (0 \leq n - m\theta \leq 1)\,, \tag{3.5}$$

where for each pair of integers n,m which satisfy $0 \leq n - m\theta \leq 1$ there exists an equivalence class of projections and we wrote an element in this class as $P_{n-m\theta}$. K_0-group $K_0(\mathcal{A}_\theta)$ is given by

$$K_0(\mathcal{A}_\theta) = \mathbb{Z} + \theta\mathbb{Z} \in \mathbb{R}\,. \tag{3.6}$$

The explicit construction of projections was given in [28] and is called Powers-Rieffel projection. Some details of this construction are reviewed in appendix A.

Let us turn to the projective modules on the non-commutative torus $\mathcal{A}_\theta$. Among various projective modules, we are interested in those which have the constant curvature connection. The explicit construction can be found in [1, 21, 29, 34] and we review this below.

Starting from a set of the rapidly decreasing functions on $\mathbb{R}$, $\phi_j(x)$, $j \in \mathbb{Z}_m$, we define the right action of U_i $(i = 1, 2)$ as follows,

$$\begin{aligned}(\phi U_1)_j(x) &= \phi_{j-1}\left(x + \frac{n}{m} - \theta\right),\\ (\phi U_2)_j(x) &= \phi_j(x) e^{2\pi i(x + jn/m)}\,.\end{aligned} \tag{3.7}$$

One may easily confirm that the operators U_1, U_2 thus defined satisfies $U_1U_2 = e^{2\pi i\theta}U_2U_1$. It shows that Schwartz space is right $\mathcal{A}_\theta$ module. It contains two integer parameters $n \in \mathbb{Z}, m \in \mathbb{Z} \geq 0$ and we denote it as $E_{n,m}$. It has the following properties,

(1) The natural "covariant derivative" for this module is given by

$$\nabla_1 = -\frac{2\pi i m}{n - m\theta}x\,, \qquad \nabla_2 = \frac{\partial}{\partial x}\,, \tag{3.8}$$

which satisfies $\nabla_i U_j = 2\pi i \delta_{ij} U_j$. It gives the constant curvature,

$$F_{12} = -F_{21} = \frac{2\pi i m}{n - m\theta}\,. \tag{3.9}$$

[6]See [21] for a more pedagogical review.

[7]A projection P is said to be unitary equivalent to another projection Q if an unitary element $u \in \mathcal{A}$ exists such that $P = u^*Qu$.

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(2) The endomorphism $\mathrm{End}_{\mathcal{A}_\theta} E_{n,m}$ is given by $\mathcal{A}_{\tilde{\theta}}$ ($\tilde{\theta} = -\frac{b-a\theta}{n-m\theta}$). It acts on $E_{n,m}$ from the left. The integers a, b are determined from n, m by the condition $an - bm = 1$. The generators of $\mathcal{A}_{\tilde{\theta}}$ act on them as

$$(Z_1\phi)_j(x) = \phi_{j-a}\left(x + \frac{1}{m}\right),$$
$$(Z_2\phi)_j(x) = \phi_j(x)e^{2\pi i(\frac{x}{n-m\theta}+\frac{j}{m})}. \tag{3.10}$$

One may easily confirm that the action of Z_i is compatible with the action of U_i since $[Z_i, U_j] = 0$. Note that if the module is free $(n, m) = (1, 0)$, then $\tilde{\theta} = \theta$.

In this way Schwartz space defines the Morita equivalence bimodule between $\mathcal{A}_\theta$ and $\mathcal{A}_{\tilde{\theta}}$. More generally, if the integers (n, m) are not coprime, then one can construct the reducible projective module $E_{n,m} = E_{n/d,m/d} \oplus E_{n/d,m/d} \oplus \cdots \oplus E_{n/d,m/d}$, where we have defined $d = \text{g.c.d}(n, m)$. In this example, $\mathcal{A}_{\tilde{\theta}}$ is Morita equivalent to $\mathrm{Mat}_d(\mathcal{A}_\theta)$ if there exist coprime integers $(n_0, m_0) \equiv (n/d, m/d)$ and (a, b) such that $\tilde{\theta} = \frac{b-a\theta}{n_0-m_0\theta}$ and $an_0 - bm_0 = -1$ [28].

It will be useful to define the explicit forms of $\mathcal{A} = \mathcal{A}_{\tilde{\theta}}$ and $\mathcal{B} = \mathcal{A}_\theta$ valued inner products in the Schwartz space. This problem is solved in much more general sense by Rieffel [29]. We write down the explicit forms for our specific example.[8] For $\phi, \psi \in S(\mathbb{R} \times \mathbb{Z}_m)$, we define,

$$\langle \phi, \psi \rangle_{\mathcal{A}} = \frac{1}{n - m\theta} \sum_{m,n} ({Z_2^{-n}} {Z_1^{-m}} \phi, \psi)_{\mathcal{A}} Z_1^m Z_2^n,$$
$$\langle \phi, \psi \rangle_{\mathcal{B}} = \sum_{m,n} (\phi, \psi {U_2^{-n}} {U_1^{-m}})_{\mathcal{B}} U_1^m U_2^n,$$
$$(\phi, \psi)_{\mathcal{B}} = (\psi, \phi)_{\mathcal{A}} = \int_{-\infty}^{\infty} dx \sum_{i\in\mathbb{Z}_m} \overline{\phi_i(x)}\psi_i(x). \tag{3.11}$$

The basic properties of the inner products (2.3) follow from the identities $(a\phi, \psi)_{\mathcal{A},\mathcal{B}} = (\phi, a^*\psi)_{\mathcal{A},\mathcal{B}}$, $(\phi\, b, \psi)_{\mathcal{A},\mathcal{B}} = (\phi, \psi\, b^*)_{\mathcal{A},\mathcal{B}}$ and so on. The proof of the associativity (2.4) is much more non trivial. We need to use the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty} f(\alpha n) = \frac{1}{\alpha} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi m}{\alpha}\right), \qquad \tilde{f}(m) = \int_{-\infty}^{\infty} e^{imx} f(x) dx. \tag{3.12}$$

After some calculations, one may confirm,

$$(\langle \phi, \psi \rangle_{\mathcal{A}} \chi)_i(x) = (\phi \langle \psi, \chi \rangle_{\mathcal{B}})_i(x) = \sum_{r,s\in\mathbb{Z}} (\phi U_1)_i^r(x) \overline{(Z_1^s \psi U_1^r)}_i(x) (Z_1^s \chi)_i(x). \tag{3.13}$$

[8]General formula for the generic Heisenberg module is not difficult to write down. One outline is sketched in [16, appendix A].

Among all connections of the projective modules, the constant curvature connection is the most useful in the physical application. One reason is that such a special connection appears as the solution of the BPS equation [21, 25]. Another reason will be given later in the arguments of the exact solution for tachyon condensation.

Finally we would like to add a comment on topological invariants. From the constant curvature connection, it is rather easy to evaluate the Chern character [25, 29] and it is known that each component becomes integer after the modification similar to Myers term [35].[9] It is of some interest to confirm this fact by using the other form of the connection/curvature. In general a connection on projective module ("Levi-Civita connection") is constructed in the form (see for example [11, 21, 34]) $\nabla_i = P_{n-m\theta} \cdot \delta_i \cdot P_{n-m\theta}$. The trace of the curvature (first Chern class) is the cyclic 2-cocycle τ_2 for $P_{n-m\theta}$. In appendix A we evaluate it by using the Powers-Rieffel projection as $P_{n-m\theta}$ and derive $\tau_2 = m$. This is of course consistent with the computation from the constant curvature connection (3.9).

3.2 Non-commutative description of BPS D-branes and T-duality

Here we discuss the spectrum and T-duality transformation rule of BPS D-branes on a two-dimensional torus with B-field flux. As is well known, there are two viewpoints for this system. One is the conventional description (commutative description) using the closed string variables. The other is by the open string variables with the non-commutative geometry [1, 2]. The results given here will be useful in the later discussions. Some general arguments can be found in [21, 37].

We investigate the dynamics of D-branes on the non-commutative two-torus $\mathcal{A}_\theta$. Since we restrict our interest to the two-dimensional case, we discuss D2-D0 bound states on the torus below. If the D2 charge and D0 charge are given by $(n, -m)$, the mass of D2-D0 bound state is determined as follows

$$\mathrm{M}^{BPS}_{(n,m)} = \frac{|n|}{\sqrt{\alpha'} g_s} \sqrt{\det\left(g + 2\pi\alpha'(B+F)\right)}\,, \tag{3.14}$$

where we have defined the gauge field strength

$$F = \frac{\mathbf{J}}{2\pi\alpha'}\frac{m}{n}\,, \qquad \mathbf{J} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.15}$$

In this formula we have taken the effect of $-m$ D0-branes into account as the shift of gauge field strength. Note that the value of n or $-m$ can be a negative integer because of T-duality. For a review of T-duality on general tori see [38]. In the following we examine transformations from a D2-brane to various D2-D0 bound states.

[9] This modification can also be regarded as "quantum effect" in terms of the quantum twisted bundle [36]. This procedure shows the relation between a bundle on the commutative torus and that on the non-commutative torus geometrically, and agrees with so-called Seiberg-Witten map [2]. We will use this map in eq. (3.22).

We define $E = g + 2\pi\alpha' B \in \mathrm{Mat}_2(\mathbb{R})$. A single D2-brane has the mass,

$$\mathrm{M}^{BPS}_{(1,0)} = \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det(E)}\,. \tag{3.16}$$

The T-duality group on $\mathbb{T}^2$ is given by $\mathrm{SO}(2,2|\mathbb{Z})$ and this acts on g, B and g_s as follows [38]

$$\begin{aligned}\tilde{E} &= \mathcal{T}(E) = (\mathcal{A}E + \mathcal{B})\,(\mathcal{C}E + \mathcal{D})^{-1}\,,\\ \tilde{g}_s &= \mathcal{T}(g_s) = g_s \det\,(\mathcal{C}E + \mathcal{D})^{-1/2}\,,\\ \mathcal{T} &= \begin{pmatrix}\mathcal{A}\mathcal{B}\\ \mathcal{C}\mathcal{D}\end{pmatrix} \in \mathrm{SO}(2,2|\mathbb{Z})\,.\end{aligned} \tag{3.17}$$

The mass $\mathrm{M}^{BPS}_{(1,0)}$ transforms into the following form

$$\begin{aligned}\mathcal{T}(\mathrm{M}^{BPS}_{(1,0)}) &= \frac{1}{\mathcal{T}(g_s)\sqrt{\alpha'}}\sqrt{\det \mathcal{T}(E)}\\ &= \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det(\mathcal{A}E + \mathcal{B})}\,.\end{aligned} \tag{3.18}$$

When the dimension of the torus is two, T-duality group can be decomposed as $\mathrm{SO}(2,2|\mathbb{Z}) \simeq \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$. One of two $\mathrm{SL}(2,\mathbb{Z})$ groups is the modular transformation of the target space $\mathbb{T}^2$. Because it preserves g_s and $\sqrt{\det(E)}$, we will not consider this part. The other $\mathrm{SL}(2,\mathbb{Z})$ is related to Morita equivalence for non-commutative $\mathbb{T}^2$ [1, 25, 31, 32] and we concentrate on this part. This $\mathrm{SL}(2,\mathbb{Z})$ transformation can be embedded into $\mathrm{SO}(2,2|\mathbb{Z})$ as follows

$$\begin{aligned}\mathrm{SL}(2,\mathbb{Z}) &\hookrightarrow \mathrm{SO}(2,2|\mathbb{Z})\,,\\ \begin{pmatrix} n & m\\ -b & -a\end{pmatrix} &\longmapsto \begin{pmatrix} n\mathbf{1} & m\mathbf{J}\\ b\mathbf{J} & -a\mathbf{1}\end{pmatrix}.\end{aligned} \tag{3.19}$$

Applying this transformation eq. (3.18) can be rewritten as

$$\mathcal{T}\left(\mathrm{M}^{BPS}_{(1,0)}\right) = \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det(n(g + 2\pi\alpha' B) + m\mathbf{J})}\,. \tag{3.20}$$

This is the same value as (3.14) and confirms that the D2-brane mass in the background $\tilde{E} = T(E)$ is equal to the mass of $(n, -m)$ D2-D0 bound state. Note also that the mass for $(n, -m)$ is equal to that for $(-n, m)$. This implies that these two configurations should be an identical state and we can restrict the integers $(n, -m)$ to $n - m\theta \geq 0$.

We translate these results into the non-commutative description. For simplicity, we fix the choice of the parameter Φ [2, 25, 32] as $B = -\Phi$. The relation between the variables in the open and closed string theories is given [2] as follows

$$\begin{aligned}B &= -\Phi = -\frac{1}{2\pi\alpha'\theta}\mathbf{J}\,, \qquad G = -(2\pi\alpha')^2 B\frac{1}{g}B\,,\\ G_s &= g_s \det(2\pi\alpha' B g^{-1})^{1/2}\,.\end{aligned} \tag{3.21}$$

This map is defined so that the mass of the single D2-brane coincides in both (open/closed) descriptions. The transformed backgrounds G and G_s are called the open string metric and the open string coupling, respectively.

When the field strength F is constant, the field strength $\hat{F}$ in the open string description[10] is given by [2]

$$\hat{F} = \frac{F}{1 + 2\pi\alpha'\theta \mathbf{J} F} \,. \tag{3.22}$$

If we apply this to the D0-D2 bound states, the flux (3.15) is transformed into

$$\hat{F} = \frac{1}{2\pi\alpha'} \frac{m}{n - m\theta} \mathbf{J} \,. \tag{3.23}$$

This is the same as the constant curvature connection on the projective module $E_{n,m}$ reviewed in the previous subsection.

Furthermore, the mass of D0-D2 bound states can also be written in terms of the open string variables as follows

$$\begin{aligned}
\mathcal{T}(\mathrm{M}_{1,0}^{BPS}) &= \mathrm{M}_{(n,m)}^{BPS} \\
&= \frac{|n|}{\sqrt{\alpha'} g_s} \sqrt{\det\left(g + 2\pi\alpha'(B+F)\right)} \\
&= \frac{|n|}{\sqrt{\alpha'} g_s} \sqrt{\det\left(g - \left(\frac{1}{\theta} - \frac{m}{n}\right) \mathbf{J}\right)} \\
&= \frac{\det(2\pi\alpha' B g^{-1})^{-1/2}}{\sqrt{\alpha'} g_s} \sqrt{\det\left(-(2\pi\alpha')^2 B g^{-1} B (n - \theta m) + \frac{n}{\theta} \mathbf{J}\right)} \\
&= \frac{n - m\theta}{\sqrt{\alpha'} G_s} \sqrt{\det(G + 2\pi\alpha'(F + \Phi))} \,.
\end{aligned} \tag{3.24}$$

Note that the rank $|n|$ of the gauge field in the commutative description is replaced with the non-integer "rank" $n - m\theta \geq 0$. In non-commutative geometry such an appearance of non integers is not surprising but very natural. Indeed this is equal to the dimension of the projective module $\dim(E_{n,m}) = n - m\theta$. As we will show later, it appears naturally in the processes of tachyon condensation. We also comment that the above formula is correct for any choice of Φ.

Finally we present an interpretation of the above results from the viewpoints of T-duality $\mathcal{T}$ on the non-commutative side. It is derived by rewriting the action $\mathcal{T}$ in (3.19) in terms of the open string variables in (3.21) [21, 25, 32],

$$\begin{aligned}
\tilde{\theta} &= \frac{b - a\theta}{n - m\theta} \,, \qquad \tilde{G}_{\mu\nu} = (n - m\theta)^2 G_{\mu\nu} \,, \\
\tilde{G}_s &= (n - m\theta) G_s \,, \qquad 2\pi\alpha'\tilde{\Phi} = (n - m\theta)^2 \left(2\pi\alpha'\Phi + \frac{m}{n - m\theta} \mathbf{J}\right) .
\end{aligned} \tag{3.25}$$

The transformation for θ is exactly the same as the Morita equivalence (3.10).

[10] Note that the curvature in mathematical conventions in section 3.1 is related to the field strength $\hat{F}$ here such that $F_{ij} = -4\pi^2 i\alpha' \hat{F}_{ij}$.

By the definition of the transformation (3.25), the map from the closed string variable to the open string variable (3.21) and the action of T-duality group on both (open /closed) sides (3.17) and (3.25) are compatible. Therefore the mass of the bound state $\mathrm{M}^{BPS}_{(n,m)}$ in the last line of (3.24) can also be obtained by acting the transformation (3.25) on $\mathrm{M}^{BPS}_{(1,0)}$ in the open string variables

$$\mathrm{M}^{BPS}_{(n,m)} = \frac{1}{\sqrt{\alpha'}\tilde{G}_s}\sqrt{\det(\tilde{G}+2\pi\alpha'\tilde{\Phi})} = \frac{n-m\theta}{\sqrt{\alpha'}G_s}\sqrt{\det(G+2\pi\alpha'(\hat{F}+\Phi))}\,. \qquad (3.26)$$

This shows that a $(n,-m)$ D-brane on the non-commutative torus $\mathcal{A}_\theta$ is a single D2-brane on $\mathcal{A}_{\tilde{\theta}}$. This result is consistent with the previous arguments in the commutative (closed string) side. Notice that the curvature $\hat{F}$ on the non-commutative torus $\mathcal{A}_\theta$ vanishes on the corresponding single D2-brane on $\mathcal{A}_{\tilde{\theta}}$ due to the shift of Φ in eq. (3.25).

The above arguments of T-duality can also be applied to non-BPS D-branes and brane-antibrane systems in the same way. We will see later that this T-duality on the non-commutative side is more directly related to Morita equivalence in the arguments of tachyon condensation.

3.3 Tachyon condensation on non-BPS D-branes

Let us discuss the tachyon condensation on non-BPS D-branes (see for example [6]) on the non-commutative torus $\mathcal{A}_\theta$. The same arguments can be applied to the bosonic string. Because any D2-D0 bound state of non-BPS D-branes can be transformed into a D2-brane, we can begin with a non-BPS D2-brane. The relation between the variables in open and closed string theories is the same as (3.21) and we continue to choose the value of Φ as $\Phi=-B$ to obtain the simplest expression. The solutions, however, do hold without any modification for general values of Φ with somewhat lengthy calculations.

On any non-BPS D-brane there exists[11] a (real scalar) tachyon field T and a gauge field A_μ. As argued in [39] the effective action of a non-BPS D2-brane can be written as

$$S = \frac{\sqrt{2}}{\sqrt{\alpha'}G_s}\int dt\,\mathrm{Tr}\left[V(T)\sqrt{\det(G+2\pi\alpha'(\hat{F}+\Phi))}\right] + O([\nabla,T],\ [\nabla,\hat{F}])\,, \quad (3.27)$$

where $[\nabla,T]$, $[\nabla,\hat{F}]$ denote the covariant derivative of the tachyon field T and the gauge field strength F; the symbol $O([\nabla,T],[\nabla,\hat{F}])$ means those terms which include one or more derivatives of T and $\hat{F}$. As we will see below our exact arguments of tachyon condensation do not depend on the detailed form of the derivative terms. Strictly speaking, we should be careful in the ordering ambiguity in the non-abelian Born-Infeld action (3.27). However our particular solutions will be defined by the

[11] In this paper we discuss the cases where the transverse scalars do not have expectation values.

projection operators. Therefore our arguments below do not depend on the detail of the ordering and we will neglect it. The factor $V(T)$ in front of the Born-Infeld term represents the tachyon potential. We normalized the value of the tachyon potential such that its value before and after the tachyon condensation into the vacuum are given by $V(1) = 1$ and $V(0) = 0$ following from Sen' conjecture [6, 7].

We use here the open string variables and therefore all the fields on the brane are regarded as operators on the non-commutative torus $\mathcal{A}_\theta$. On the non-BPS D-brane, the tachyon and the gauge field belong to the adjoint representation of the gauge group. In the language of the non-commutative geometry they are expressed as elements in the endomorphism $\mathrm{End}_{\mathcal{A}^{(0)}} E$ of the projective module E. Before the tachyon condensation, the projective module should represent the original D2-brane itself $E_{1,0} = \mathcal{A}_\theta$. After the tachyon condensation, it should be projected into a non-trivial projective module of $\mathcal{A}_\theta$ which we investigate below.

We would like to solve the equation of motion of the tachyon field by imposing several assumptions. Sufficient conditions are

$$\frac{\partial V(T)}{\partial T} = 0\,, \qquad [\nabla, T] = 0\,. \tag{3.28}$$

The first equation is equivalent to the equation of motion if we take the large-B limit as discussed in [8, 9, 10]. The solutions to this equation are given by the projections in the non-commutative torus algebra $\mathcal{A}_\theta$. The projections in $\mathcal{A}_\theta$ are classified by the values of trace as in (3.5) and we write $T = P_{n-m\theta}$. Next we examine the second condition $[\nabla, T] = 0$. This is satisfied[12] if we use the connection $\nabla_i = P_{n-m\theta}(\delta_i + A_i)P_{n-m\theta}$. This is the standard definition of a connection for projective modules [21]. In the same way as in [20] we have to regard the covariant derivative as a whole as a dynamical variable. It defines the projective module $E = P_{n-m\theta}E_{1,0}$ and the endomorphism $\mathrm{End}_{\mathcal{A}_\theta} E$ is given by $P_{n-m\theta}\mathcal{A}_\theta P_{n-m\theta}$. The projective module E is twisted and its first Chern class (or cyclic 2-cocycle) is given by $\tau_2(P_{n-m\theta}) = m$ as we explain in appendix A.

Now let us turn to the equation for the gauge field $[\nabla_i, \hat{F}_{ij}] = 0$. It is satisfied if the field strength is proportional to $P_{n-m\theta}$. With this form it is clear that it also minimizes the DBI action. In a sense, $P_{n-m\theta}$ can be regarded as the identity in the algebra $\mathrm{End}_{\mathcal{A}_\theta} E$. The field strength which is proportional to the projector should be regarded as constant curvature reviewed in the section 3.2.

One may prove that for any projection of the type $P_{n-m\theta}$ the projective module of the form $E = P_{n-m\theta}E_{1,0} = P_{n-m\theta}\mathcal{A}_\theta$ has a constant curvature connection. As we saw in the subsection 3.1, for every n, m one can construct a constant curvature

[12] One can also satisfy $[\nabla, T] = 0$ if the connection acts only on the projective module $E' = (1 - P_{n-m\theta})E_{1,0}$. However a little analysis shows that this is not consistent with the descent relation [40] and does not have a expected tension. Therefore we believe that this does not correspond to physical solutions and neglect these in this paper.

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connection as the Heisenberg projective module. Because it is projective, it should be written as $\tilde{E} = \tilde{P}_{n-m\theta}\mathcal{A}_\theta^N$ for a certain projection $\tilde{P}_{n-m\theta}$ in $\mathrm{Mat}_N(\mathcal{A}_\theta)$. Because any projection in $\mathrm{Mat}_N(\mathcal{A}_\theta)$ for a given value of trace belongs to the same K-theory class [28], we can change $\tilde{P}_{n-m\theta}$ into $P_{n-m\theta} \in \mathcal{A}_\theta$ via an unitary transformation. The transformed projective module $E = P_{n-m\theta}\mathcal{A}_\theta$ also possesses the induced constant curvature connection.

In this way we have found exact solutions of the equation of motion derived from (3.27).

$$T = P_{n-m\theta}\,, \qquad \hat{F} = \frac{1}{2\pi\alpha'}\frac{m}{n-m\theta}P_{n-m\theta}\,\mathbf{J}\,. \tag{3.29}$$

Here we represent these fields as elements in the "large" algebra $\mathcal{A}_\theta$. In the small algebra $\mathrm{End}_{\mathcal{A}_\theta} E$, both T and A_i are proportional to the identity. Unlike the Moyal plane case, the small algebra is Morita equivalent to $\mathcal{A}_\theta$ and can be rewritten as

$$\begin{aligned} \mathrm{End}_{\mathcal{A}_\theta} E &= P_{n-m\theta}\mathcal{A}_\theta P_{n-m\theta} = \mathrm{Mat}_d(\mathcal{A}_{\tilde{\theta}})\,, \\ &\quad \left(\tilde{\theta} = \frac{b-a\theta}{n_0-m_0\theta} \quad \text{s.t.} \;\; an_0 - bm_0 = -1\right), \end{aligned} \tag{3.30}$$

where we have defined $d = \text{g.c.d.}(n,m)$ and $(n,m) = d(n_0,m_0)$.

We proceed to discuss what will be generated via the tachyon condensation (3.29). The mass of this excitation can be evaluated by neglecting the derivative terms because of $[\nabla, T] = [\nabla, F] = 0$,

$$\begin{aligned} M_{(n,m)} &= \frac{\sqrt{2}}{\sqrt{\alpha'}G_s}\,\mathrm{Tr}\left[P_{n-m\theta}\sqrt{\det(G + 2\pi\alpha'(\hat{F}+\Phi))}\right] \\ &= \frac{\sqrt{2}|n|}{\sqrt{\alpha'}g_s}\sqrt{\det\left(g + 2\pi\alpha'(B+F)\right)}\,, \end{aligned} \tag{3.31}$$

where the flux F is given by (3.15). The factor $\sqrt{2}$ is peculiar to non-BPS D-branes. The above calculation can be done in the same way as in (3.24). The dimension of the projective module naturally appears here as the trace of the projection. It is interesting that the tachyon condensation on unstable D-brane systems gives an explicit physical realization of the mathematically fundamental relation between projective modules and projections.

If we assume $g_{ij} = R^2\delta_{ij}$ to make discussion clearer, the mass reduces to

$$M_{(n,m)} = \frac{\sqrt{2}}{\sqrt{\alpha'}g_s}\sqrt{n^2R^2 + \left(\frac{n}{\theta} - m\right)^2}\,. \tag{3.32}$$

It explicitly shows that the resulting state is a bound state of n non-BPS D2-branes and $(-m)$ non-BPS D0-branes. We comment that our arguments of tachyon condensation naturally derive the fact that $-m$ or n can be negative which is consistent with the result in the previous subsection.[13] In this way we obtain all kinds of D2-D0 bound states via tachyon condensation and thus our results are consistent with T-duality.

If we take the large-B/g limit, the mass spectrum is proportional to $\dim(E) = n - m\theta$ as can be seen from (3.32) and it is dense in $\mathbb{R}$. This means that there exists an excitation of which energy is arbitrary small. In [14] we investigated the tachyon condensation in this limit and suggested that it leads to an instability. It means that any projection $P_{n-m\theta}$ can be divided into infinite numbers of mutually orthogonal smaller projections while the total value of the trace is preserved. This was proved by investigating an explicit representation of projections.

The argument can be simplified as follows. We start from a projection $P_{n-m\theta}$ in $\mathcal{A}_\theta$. It can be regarded as the identity in the small algebra $\mathrm{Mat}_d(\mathcal{A}_{\tilde{\theta}}) = P_{n-m\theta}\mathcal{A}_\theta P_{n-m\theta}$. One can find another projection Q in the small algebra. The original projection is decomposed into two mutually orthogonal projections,

$$P_{n-m\theta} = (1-Q)P_{n-m\theta} + QP_{n-m\theta}\,. \tag{3.33}$$

One may continue this operation repeatedly to give the infinitely small mutually orthogonal projectors.

We would like to claim that such an instability does not appear for finite B/g. As can easily be seen from the mass formula (3.32), the bound state can be divided into $d = \text{g.c.d}\ (n,m)$ pieces but is not divided further. It means the instability cannot appear. We may interpret it from our exact tachyon solution. The key point is the requirement of the constant field strength in (3.29) which is absent in large-B/g limit. It is permitted to be divided into only d mutually orthogonal parts, even though the tachyon field in (3.29) can be divided into infinitely many pieces. We conclude that for finite B/g the bound states are all stable if $\text{g.c.d}\ (n,m) = 1$.

The corresponding projective module $E_{n,m} = P_{n-m\theta}E_{1,0}$ can be written as a direct sum of d projective modules of the same type as we saw in section 3.1. As shown in [41] (see also [21]), the moduli space of constant curvature connection is given by the symmetric product of d copies of a two-torus $(\mathbb{T}^2)^d/S_d$. This is actually the same as the physical moduli space of the solutions of (3.27) up to gauge transformation. The freedom of unitary transformation of the tachyon field is absorbed in the gauge transformation $T \to UTU^*$, $U \in \mathcal{A}_\theta$ and what remains is only the moduli space of constant curvature connection for the projective module $E_{n,m} = P_{n-m\theta}E_{1,0}$. We note that this is consistent with the following physical intuition. The moduli space

[13] One may ask the physical interpretation for the negative n. Assume that n is negative and thus $-m$ is positive. Since D0-branes with B-field on a torus generate a (non-BPS) D2-brane, we can say that the negative n means the annihilation of these induced non-BPS D2-branes with the $|n|$ non-BPS D2-branes.

of a bound state for coprime (n, m) parameterizes the transverse coordinate for a D0-brane, namely $\mathbb{T}^2$. It is then obvious that the moduli space of the bound states of d D0-branes should be $(\mathbb{T}^2)^d/S_d$.

The fluctuations of the gauge and tachyon fields around the solution (3.29) belong to $\mathrm{Mat}_d(\mathcal{A}_{\tilde{\theta}})$. This is physically interpreted as the gauge group on the brane is $\mathrm{U}(d)$.

Up to now we concentrate on the tachyon condensation on a single D2-brane. We may, of course, start from plural or even infinitely many D2-branes. In such a situation, we can obtain arbitrary projective modules $E_{n,m}$ from the projection in $\mathrm{Mat}_\infty(\mathcal{A}_\theta)$.

It is also important to note the relation between tachyon condensation on the above non-commutative torus and that on the non-compact flat plane (Moyal plane). The exact solution for the latter [20] can be rewritten in our convention as follows,

$$T = P_l\,, \qquad \hat{F} = \frac{1}{\Theta} P_l\, \mathbf{J}\,, \qquad (l \in \mathbb{Z} \geq 0)\,, \tag{3.34}$$

where the noncommutativity Θ is defined as $[x^1, x^2] = i\Theta$. The level-l projection P_l in the Moyal plane algebra is given by $\sum_{k=0}^{l-1} |k\rangle\langle k|$ and it corresponds to the generation of l D0-branes. To obtain such situation from the torus, we need to take the large radius limit or equivalently the small θ limit. In such a situation, the value of n is restricted to 0 or 1 which is consistent with the result (3.34). We note that in this limit one cannot take $B/g \to \infty$ limit and thus the instability does not occur.

Solution generating technique and Morita equivalence We have seen the exact description of tachyon condensation is characterized by the constant curvature connection. It is interesting to ask what is the solution generating method which relates various solutions. In the Moyal plane, the exact solutions for tachyon condensation were constructed in [20] by using the shift operator $S = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$. We would like to find the analogous transformation on the two-torus $\mathcal{A}_\theta$.

In our case we have to be careful since the corresponding operator in general interpolates the different C^*-algebras. Namely after the tachyon condensation the original free module $E_{1,0} = \mathcal{A}_\theta$ (a D2-brane) is changed into the twisted projective module $E_{n,m}$ (a D2-D0 bound state). As we have seen in section 2, the transformation between these two solutions should be identified with the tensor action of the Morita equivalence bimodule S. It depends on the integers n, m and satisfies

$$E_{n,m} = S \otimes_{\mathcal{A}_\theta} E_{1,0}\,. \tag{3.35}$$

This maps the endomorphism $\mathrm{End}_{\mathcal{A}_\theta} E_{1,0} = \mathcal{A}_\theta$ into $\mathrm{End}_{\mathcal{A}_\theta} E_{n,m} = \mathrm{Mat}_d(\mathcal{A}_{\tilde{\theta}})$. Physically this induces the transformation of the world-volume field theories and this gives an explicit realization of the descent relation [40].

In order to describe the exact solution, the projective module $E_{n,m}$ should have the constant curvature. In other words, we have to impose on the Morita equivalence $\mathcal{A}_{\tilde{\theta}}$-$\mathcal{A}_\theta$ bimodule that S should keep this additional constraint. Actually it just fits the definition of the *gauge Morita equivalence bimodule* discussed in [21, 25]. We claim that this is the analogue of the shift operator on the non-commutative torus.

We mention that T-duality transformation can also be represented by the gauge Morita equivalence bimodules as argued in [21, 25, 32]. On a non-commutative torus $\mathcal{A}_\theta$ there is a one-to-one correspondence between $\mathcal{A}_\theta$-modules and the solutions found in (3.29). They correspond to the projective module $E_{n,m}$ and take their values in $\mathrm{End}_{\mathcal{A}_\theta} E_{n,m}$. If we perform T-duality so that a $(n,-m)$ brane is transformed into d D2-branes (or equally applying the Morita equivalence $\mathcal{A}_\theta \sim \mathcal{A}_{\tilde{\theta}}$), the projective module $E_{n,m}$ is changed into the free module $\tilde{E}_{d,0}$ over the algebra $\mathcal{A}_{\tilde{\theta}}$. Such T-duality transformation can be constructed by $\mathcal{A}_\theta$-$\mathcal{A}_{\tilde{\theta}}$ gauge Morita equivalence bimodule X

$$\tilde{E}_{d,0} = E_{n,m} \otimes_{\mathcal{A}_\theta} X\,. \tag{3.36}$$

The endomorphism $\mathrm{End}_{\mathcal{A}_\theta} E_{n,m} = \mathrm{End}_{\mathcal{A}_{\tilde{\theta}}} \tilde{E}_{d,0}$ acts on $\tilde{E}_{d,0}$ from the left in (3.36). Especially, the solution $T = \mathbf{1} \in \mathrm{End}_{\mathcal{A}_\theta} E_{n,m}$ is the identity in $\mathrm{End}_{\mathcal{A}_{\tilde{\theta}}} \tilde{E}_{d,0}$. On the other hand, the constant curvature $\hat{F} = \frac{1}{2\pi\alpha'}\frac{m}{n-m\theta}\mathbf{J} \in \mathrm{End}_{\mathcal{A}_\theta} E_{n,m}$ is translated to $0 \in \mathrm{End}_{\mathcal{A}_{\tilde{\theta}}} \tilde{E}_{d,0}$, because $\tilde{E}_{d,0}$ is the free module over $\mathcal{A}_{\tilde{\theta}}$. The difference between the values of the constant curvatures comes from the constant curvature of X. This is equivalent to the shift of the field Φ by T-duality transformation in (3.25). These imply that for general non-commutative algebras the exact solution for tachyon condensation can be generated in the same way if there exist gauge Morita equivalence bimodules.

3.4 Tachyon condensation on brane-antibrane systems

Brane-antibrane systems are more complicated and intriguing than non-BPS D-branes from the viewpoint not only of string theory but also of the non-commutative geometry. The crucial difference from non-BPS D-branes is that the tachyon field becomes complex and belongs to a Morita equivalence $\mathcal{A}$-$\mathcal{B}$ bimodule (we write this as X), where $\mathcal{A}$ and $\mathcal{B}$ are the algebras of the brane and the antibrane as we discussed in section 2. If we define E and F as the projective modules which represent the brane and the antibrane, respectively and define $\mathcal{A}^{(0)}$ as the non-commutative base space, the algebras $\mathcal{A}, \mathcal{B}$ are given by $\mathcal{A} = \mathrm{End}_{\mathcal{A}^{(0)}} E,\ \mathcal{B} = \mathrm{End}_{\mathcal{A}^{(0)}} F$. Remember that a Morita equivalence bimodule X is defined by the bimodule which possesses two types of inner products $\langle\ ,\ \rangle_{\mathcal{A}},\ \langle\ ,\ \rangle_{\mathcal{B}}$ and satisfies the conditions (2.3) and (2.4).

As in the previous subsection we use the example of non-commutative two-tori and consider only D2-D0 bound states. We assume the original brane-antibrane system is made of a $(n_1, -m_1)$ brane and a $(n_2, -m_2)$ antibrane. For simplicity we consider only the case where the pairs of integers $(n_1, -m_1)$ and $(n_2, -m_2)$ are coprime. A pair of integers $(n, -m)$ denotes the indices of D2-D0 bound states or equivalently those of the corresponding projective module $E_{n,m}$. Note that if one specifies (n, m) such that $n - m\theta \geq 0$, then there are two types of D-branes, that is, branes and antibrane. In our examples of the non-commutative torus the fundamental algebra $\mathcal{A}^{(0)}$ is given by $\mathcal{A}^{(0)} = \mathcal{A}_\theta$, where θ is represented in terms of closed string variables as in (3.21). The algebras $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_{\theta_1}\,, \qquad \theta_1 = \frac{b_1 - a_1\theta}{n_1 - m_1\theta}\,, \qquad (a_1 n_1 - b_1 m_1 = -1)\,, \\ \mathcal{B} &= \mathcal{A}_{\theta_2}\,, \qquad \theta_2 = \frac{b_2 - a_2\theta}{n_2 - m_2\theta}\,, \qquad (a_2 n_2 - b_2 m_2 = -1)\,. \end{aligned} \tag{3.37}$$

The tachyon field T belongs to a $\mathcal{A}_{\theta_1}$-$\mathcal{A}_{\theta_2}$ bimodule. There are also the gauge fields on the brane and the antibrane. We denote these as $A^{(1)}$ and $A^{(2)}$. These fields belong to $\mathcal{A} = \mathcal{A}_{\theta_1}$ and $\mathcal{B} = \mathcal{A}_{\theta_2}$, respectively. The covariant derivative of the tachyon field is given by the connection for a bimodule X (see for example [21, 25]) specified by the requirement

$$\begin{aligned} \nabla_X(aT) &= \delta_{\mathcal{A}}(a)T + a\nabla_X T\,, \qquad (\forall a \in \mathcal{A})\,, \\ \nabla_X(Tb) &= (\nabla_X T)b + T\delta_{\mathcal{B}}(b)\,, \qquad (\forall b \in \mathcal{B})\,, \end{aligned} \tag{3.38}$$

where $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ denote the derivation in $\mathcal{A}$ and $\mathcal{B}$, respectively. We also use the covariant derivative of field strengths $\hat{F}^{(1)}$ and $\hat{F}^{(2)}$. Each of them is given by the commutator with a connection for the algebra $\mathcal{A}$ or $\mathcal{B}$ as in the previous subsection.

Now we have prepared to discuss the tachyon condensation on non-commutative tori. The effective action for a D2$-\overline{\text{D2}}$ with gauge flux was already computed in [24] using the boundary string field theory [42, 43, 44]. Applying this to our system on a non-commutative torus the result is given by

$$\begin{aligned} S = {} & \frac{1}{\sqrt{\alpha'}G_s}\int dt\, \mathrm{Tr}_{\mathcal{A}}\left[V\left(\langle T, T\rangle_{\mathcal{A}}\right)\sqrt{\det(G + 2\pi\alpha'(\hat{F}^{(1)} + \Phi))}\right] + \\ & + \frac{1}{\sqrt{\alpha'}G_s}\int dt\, \mathrm{Tr}_{\mathcal{B}}\left[V\left(\langle T, T\rangle_{\mathcal{B}}\right)\sqrt{\det(G + 2\pi\alpha'(\hat{F}^{(2)} + \Phi))}\right] + \\ & + O(\nabla_X T,\ [\nabla_{\mathcal{A}}, \hat{F}^{(1)}],\ [\nabla_{\mathcal{B}}, \hat{F}^{(2)}])\,, \end{aligned} \tag{3.39}$$

where $O(\nabla_X T, \ldots)$ denotes the derivative terms. We defined the traces for $\mathcal{A}$ and $\mathcal{B}$ by embedding these algebras in $\mathrm{Mat}_N(\mathcal{A}^{(0)})$ for a sufficient large integer N. These satisfy the following relation [28, 29]

$$\mathrm{Tr}_{\mathcal{A}}\langle T_1, T_2\rangle = \mathrm{Tr}_{\mathcal{B}}\langle T_2, T_1\rangle\,, \tag{3.40}$$

and are normalized by $\mathrm{Tr}_{\mathrm{Mat}_N(\mathcal{A}^{(0)})}1 = N$.

We would like to solve the equation of motion for the action (3.39). Below we give solutions by imposing an ansatz similar to the previous subsection. We assume the existence of the partial isometry-like equation [19, 20, 30] for the tachyon field (2.10)

$$\langle T, T\rangle_{\mathcal{A}}T = T\,, \qquad T\langle T, T\rangle_{\mathcal{B}} = T\,. \tag{3.41}$$

Note that these two equations are equivalent thanks to the relation (2.4). The solutions to this equation give the stationary points of the tachyon potential $V(\langle T,T\rangle_{\mathcal{A}})$ and $V(\langle T,T\rangle_{\mathcal{B}})$. To make exact solutions for finite B/g, we should take account of the gauge fields. It is easy to see the equation of motions for the tachyon T and gauge fields ($A^{(1)}$, $A^{(2)}$) are satisfied if we require (3.41) and

$$\nabla_X T = 0\,, \tag{3.42}$$
$$[\nabla_{\mathcal{A}}, \hat{F}^{(1)}] = 0\,, \qquad [\nabla_{\mathcal{B}}, \hat{F}^{(2)}] = 0\,, \tag{3.43}$$

are met. Since (3.41) is equivalent to the statement that $\langle T,T\rangle_{\mathcal{A}} \in \mathcal{A}$ and $\langle T,T\rangle_{\mathcal{B}} \in \mathcal{B}$ are both projections as explained in section 2, we can write these as follows

$$\langle T,T\rangle_{\mathcal{A}} = 1 - P_{\alpha+\beta\theta_1} (\equiv 1 - P_1)\,, \qquad \langle T,T\rangle_{\mathcal{B}} = 1 - P_{\gamma+\delta\theta_2} (\equiv 1 - P_2)\,. \tag{3.44}$$

Using the relation (3.40) we obtain the constraint,

$$n_1(1-\alpha) - b_1\beta = n_2(1-\gamma) - b_2\delta\,, \qquad m_1(1-\alpha) - a_1\beta = m_2(1-\gamma) - a_2\delta\,. \tag{3.45}$$

They determine γ and δ in terms of α and β. The tachyon field which condensates as in (3.44) belongs to the bimodule $(1-P_1)\cdot X \cdot (1-P_2)$. For this tachyon field the potential is evaluated as

$$V(\langle T,T\rangle_{\mathcal{A}}) = P_{\alpha+\beta\theta_1}\,, \qquad V(\langle T,T\rangle_{\mathcal{B}}) = P_{\gamma+\delta\theta_2}\,. \tag{3.46}$$

This is because we use the convention that the original brane-antibrane system corresponds to $T = 0$ and that the vanishing of brane and antibrane corresponds to $\langle T,T\rangle_{\mathcal{A}} = 1$ and $\langle T,T\rangle_{\mathcal{B}} = 1$, respectively.[14]

Let us turn to the next equation (3.42). This is satisfied if the gauge fields belong to

$$A^{(1)} \in P_1\mathcal{A}P_1\,, \qquad A^{(2)} \in P_2\mathcal{B}P_2\,. \tag{3.47}$$

The last equation (3.43) is solved if we assume that both of the gauge fields have constant curvatures

$$\begin{aligned} \hat{F}^{(1)} &= \frac{1}{2\pi\alpha'}\frac{\beta a_1 + m_1\alpha}{n_1\alpha + b_1\beta - (\beta a_1 + m_1\alpha)\theta}\mathbf{1} \in P_1\mathcal{A}P_1\,, \\ \hat{F}^{(2)} &= \frac{1}{2\pi\alpha'}\frac{\delta a_2 + m_2\gamma}{n_2\gamma + b_2\delta - (\delta a_2 + m_2\gamma)\theta}\mathbf{1} \in P_2\mathcal{B}P_2\,. \end{aligned} \tag{3.48}$$

It finishes our calculation of the physical solutions of the tachyon condensation. Our result does not depend on the detailed form of the derivative term because of (3.42) and (3.43).

[14]Notice that this is the opposite convention to the tachyon potential of non-BPS D-branes in section 3.3.

We evaluate mass spectrum of these classical solutions,

$$
\begin{aligned}
M &= \frac{1}{\sqrt{\alpha'}G_s}\,\mathrm{Tr}_{\mathcal{A}}\left[P_{\alpha+\beta\theta_1}\sqrt{\det(G+2\pi\alpha'(\hat{F}^{(1)}+\Phi))}\right]+\\
&\quad+\frac{1}{\sqrt{\alpha'}G_s}\,\mathrm{Tr}_{\mathcal{B}}\left[P_{\gamma+\delta\theta_2}\sqrt{\det(G+2\pi\alpha'(\hat{F}^{(2)}+\Phi))}\right]\\
&= \frac{|N_1|}{\sqrt{\alpha'}g_s}\sqrt{\det(g+2\pi\alpha'(B+F^{(1)}))}+\\
&\quad+\frac{|N_2|}{\sqrt{\alpha'}g_s}\sqrt{\det(g+2\pi\alpha'(B+F^{(2)}))}\,.
\end{aligned}
\tag{3.49}
$$

Here we have defined the flux as follows

$$
F^{(1)}=\frac{\mathbf{J}}{2\pi\alpha'}\frac{M_1}{N_1}\,,\qquad F^{(2)}=\frac{\mathbf{J}}{2\pi\alpha'}\frac{M_2}{N_2}\,,
\tag{3.50}
$$

where integers M_1, N_1, M_2 and N_2 are given by

$$
\begin{aligned}
N_1 &= \alpha n_1+\beta b_1\,, \qquad M_1=\beta a_1+\alpha m_1\,,\\
N_2 &= \gamma n_2+\delta b_2\,, \qquad M_2=\delta a_2+\gamma m_2\,.
\end{aligned}
\tag{3.51}
$$

We find that the products of the tachyon condensation are identified with a $(N_1,-M_1)$ brane and a $(N_2,-M_2)$ anti-brane. If these integers are not coprime, each bound state can be divided into several parts as before. Note that what is produced after the tachyon condensation depends on the projections (3.46). The original brane-antibrane system corresponds to $\alpha=\gamma=1$, $\beta=\delta=0$. If one assumes that the dimension of the projective module E is larger than that of F, then the tachyon field with $\gamma=0,\delta=0$ gives the maximal condensation and this will produce a $(n_1-n_2,-m_1+m_2)$ brane. In the opposite case the tachyon field with $\alpha=0$, $\beta=0$ will generate a $(n_2-n_1,-m_2+m_1)$ anti-brane.

For general decay modes the differences of the D2-brane charge (N_1-N_2) and the D0-brane charge (M_2-M_1) are preserved as follows

$$
N_1-N_2=n_1-n_2\,,\qquad M_2-M_1=m_2-m_1\,.
\tag{3.52}
$$

The charge conservation can also be discussed in the framework of operator algebra K_0-group $K_0(\mathcal{A}_\theta)$. If one would like to consider the K_0-group of non-commutative torus, the Chern character [34] gives enough information [29]. In the brane-antibrane system which corresponds to the pair of projective modules $(E,F)\in K_0(\mathcal{A}_\theta)$ the K-theory charge is given by the difference

$$
\begin{aligned}
\mathrm{ch}(E)-\mathrm{ch}(F) &= \mathrm{Tr}_{\mathcal{A}}\exp(2\pi\alpha'\hat{F}^{(1)})-\mathrm{Tr}_{\mathcal{B}}\exp(2\pi\alpha'\hat{F}^{(2)})\\
&= (n_1-m_1\theta-n_2+m_2\theta)+(m_1-m_2)dx^1dx^2\,,
\end{aligned}
\tag{3.53}
$$

where dx^1dx^2 is the two form along the two-torus. It is known that the RR-couplings on a brane-antibrane system can be written by using K^0-type superconnection [22, 23, 24]. Applying this idea to our examples we obtain the following RR-couplings

$$S_{\mathrm{RR}} \sim \int C_{\mathrm{RR}} \wedge \Big[\mathrm{Tr}_{\mathcal{A}}\, V\left(\langle T,T\rangle_{\mathcal{A}}\right) \exp(2\pi\alpha' \hat{F}^{(1)}) - \\ - \mathrm{Tr}_{\mathcal{B}}\, V\left(\langle T,T\rangle_{\mathcal{B}}\right) \exp(2\pi\alpha' \hat{F}^{(2)})\Big]\,. \qquad (3.54)$$

Note that in our non-commutative description the derivative of the tachyon field is always zero and does not contribute. Also notice that the potential in the above can be regarded as the identities in the algebras of gauge fields $\mathrm{End}_{\mathcal{A}_\theta} E$ and $\mathrm{End}_{\mathcal{A}_\theta} F$ because of (3.46). Thus the conservation of RR-charge (3.52) is equal to that of operator algebra K-theory charge and therefore this gives a further support to the relation between D-brane charge and K-theory [4, 5, 13]. The intriguing characteristic that the above "non-commutative Chern character" depends on θ will correspond to the physical fact that D0-branes in the B-field background generate D2-brane charge [45]. Therefore it will also be interesting to clarify the relation between the RR-couplings for non-abelian transverse scalars [35, 24] and the above RR-couplings on various non-commutative tori [45] (see also [46] for the Moyal plane).

4. Conclusions

We first discussed open string theory in general non-commutative background. We considered a general framework to handle open strings and D-branes in a unified way by utilizing the Morita equivalence. In particular we proposed the equation which defines non-commutative solitons on general brane-antibrane systems.

From this viewpoint we have examined the exact solutions on non-commutative tori in tachyonic systems. For non-BPS D-branes the solutions are given by the tachyon field which is proportional to the projection and the gauge field with a constant curvature. This respects the one-to-one correspondence between a projection and a projective module. We have also shown that these solutions can be generated by employing the (gauge) Morita equivalence. Our exact description of tachyon condensation including the gauge field solves for finite B/g the previously observed instability problem.

More complicated and thus more intriguing examples are brane-antibrane systems. In these cases the tachyon field belongs to the Morita equivalence bimodule and we can impose the partial isometry-like relation instead of the equation of motion. We can construct the exact solutions and determine the decay products. We find the RR-charges of brane-antibrane systems can be represented by the superconnection-like extension of the Connes's Chern character and check that these charges conserved in the process of tachyon condensation. This also verifies the fact that the D-brane charge is classified by the operator algebra K-theory.

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Note added. After completing our calculations, we noticed the preprint [47] on the net which has some overlaps with our results in section 3.3.

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A. Powers-rieffel projections and cyclic cocyles

Here we give a review of Powers-Rieffel projections [28] on two-dimensional non-commutative tori and of the calculations of their topological charges [3, 34]. We also mention some other projections constructed in [14].

First let us consider projections in any C^*-algebra $\mathcal{A}$ and assume that there exist a trace $\mathrm{Tr} : \mathcal{A} \to \mathbf{C}$ and derivations $\delta_i : \mathcal{A} \to \mathcal{A}$ such that $\delta_i(ab) = \delta_i(a)b + a\delta_i(b)$. We also normalize the trace as $\mathrm{Tr}(\mathbf{1}) = 1$. The index i corresponds to the basis of the derivations.[15] A projection $p \in \mathcal{A}$ is defined to be a self-adjoint idempotent $p^* = p = p^2$. The Connes's Chern Character is defined as the exponential of the curvature $F \in \mathrm{End}_{\mathcal{A}} E$ of projective module E [34]:

$$\mathrm{ch}(E) = \mathrm{Tr}\exp\left(\frac{F}{2\pi i}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\, \tau_{2k}(F, F, \ldots, F)\,. \tag{A.1}$$

The each term $\tau_{2k}(F, F, \ldots, F)$ of the above expansion represents the contribution which is proportional to F^k. Note that τ_0 is equal to the trace of identity and it gives the dimension $\dim(E)$ of the projective module.

Since any projective module E is described as $P \cdot \mathcal{A}^N$ using a projection P in $\mathrm{Mat}_N(\mathcal{A})$ for a sufficiently large integer N, one can rewrite the curvature in terms of the projection. More explicitly one can choose a connection of E as $\nabla_i = P \cdot \delta_i \cdot P$ [21, 34]. Note that the topological quantity such as the Chern character does not depend on the choice of the connection. Then the curvature $F \in \mathrm{End}_{\mathcal{A}} E$ is expressed as

$$F_{ij} = [P\delta_i P,\ P\delta_j P] = P(\delta_i P)(\delta_j P) - P(\delta_i P)(\delta_j P) + P(\delta_i\delta_j - \delta_j\delta_i)P\,. \tag{A.2}$$

[15] More precisely, the derivation is defined as an action of a Lie group G on the algebra $\mathcal{A}$ [34]. Then the basis of the derivation can be said as those of the Lie algebra.

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Because the trace in $\mathrm{End}_{\mathcal{A}} E = P \cdot \mathrm{Mat}_N(\mathcal{A}) \cdot P$ is naturally induced from the trace in $\mathrm{Mat}_N(\mathcal{A})$ normalized as $\mathrm{Tr}(\mathbf{1}) = N$, one can always calculate the Chern character (A.1) if the projection P is given. Thus we write the $2k$-th part by $\tau_{2k}(P)$. This can be regarded as a cyclic $2k$-cocycle where the projections are substituted.

For example, the cyclic 0-cocycle is given by

$$\tau_0(P) = \mathrm{Tr}(P) = \dim(E)\,. \tag{A.3}$$

Now let us investigate a explicit examples of cyclic cocycles. First we consider a flat two-dimensional plane (Moyal plane) algebra. We employ an operator representation and define the non-commutative coordinate (x^1, x^2) as $[x^1, x^2] = i\Theta$. Further we define the creation and annihilation operator $a^\dagger = \frac{1}{\sqrt{2\Theta}}(x^1 - ix^2)$ and $a = \frac{1}{\sqrt{2\Theta}}(x^1 + ix^2)$ such that they satisfy $[a, a^\dagger] = 1$. Using this one can define the basis of the Hilbert space as the familiar n-number state $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$. Then the algebra is expanded by $|n\rangle\langle m|$ $(n, m \geq 0)$. Let us consider the projection $P_n = \sum_{k=0}^{n-1} |k\rangle\langle k|$ for a finite integer n. The trace of this operator is given by $\mathrm{Tr}(P_n) = 2\pi\Theta \cdot n$, where we have normalized the trace so that it is equal to the integration $\int dx^1 dx^2$ in the c-function representation. Now it is straightforward to calculate $\tau_2(P_n)$ since the derivations δ_1, δ_2 along the coordinate x^1, x^2 are given by

$$\delta_1 = \frac{i}{\Theta}[x^2,\ ]\,, \qquad \delta_2 = -\frac{i}{\Theta}[x^1,\ ]\,. \tag{A.4}$$

Thus we obtain the result as follows

$$\begin{aligned}
\tau_2(P_n) &= \frac{1}{2\pi\Theta}\,\mathrm{Tr}\left[P_n[a^\dagger, P_n][a, P_n] - P_n[a, P_n][a^\dagger, P_n]\right] \\
&= \frac{1}{2\pi\Theta}\,\mathrm{Tr}\left[P_n + P_n a^\dagger P_n a P_n - P_n a P_n a^\dagger P_n\right] \\
&= n\,.
\end{aligned} \tag{A.5}$$

Therefore the value of $\tau_2(P_n)$ is quantized and is positive. Note that this value is equal to the first Chern class of the projective module and therefore should be quantized. Then one may ask if one can obtain negative integers? The answer is yes and one can construct the corresponding projection as $1 - P_n$. In this case we obtain the value $\tau_2(1 - P_n) = -n$.

After this elementary example, let us turn to the two-dimensional non-commutative torus $\mathcal{A}_\theta$. We assume θ is irrational because for rational θ there are finite-dimensional representations of the algebra and the calculations are simplified (see also [14, 16, 17, 18]).

As we have mentioned in section 3.1, the projections in $\mathcal{A}_\theta$ are generally characterized by their values of trace as in eq. (3.5). The explicit construction of projections (Powers-Rieffel projection) was given in [28] and let us review this below.

Because U_1 and U_2 do not commute ($U_1U_2 = U_2U_1e^{2\pi i\theta}$), we can diagonalize only U_2 and define c-number x^2 as $U_2 = e^{2\pi i x^2}$. Thus we obtain the following (infinite-dimensional) representation of $\mathcal{A}_\theta$

$$\begin{aligned} U_1|x^2\rangle &= |x^2+\theta\rangle\,, \\ U_2|x^2\rangle &= e^{2\pi i x^2}|x^2\rangle\,. \end{aligned} \tag{A.6}$$

Then the trace of an element $a \in \mathcal{A}_\theta$ is given by

$$\mathrm{Tr}(a) = \int_0^1 dx^2 \langle x^2|a|x^2\rangle\,. \tag{A.7}$$

In order to find explicit projections P we assume the following form

$$P = U_1^*\,(g(U_2))^* + f(U_2) + g(U_2)U_1\,. \tag{A.8}$$

As we will see below, one can construct a projection for each n, m even under this restriction. Acting on the position space $|x^2\rangle$, we require $P^2|x^2\rangle = P|x^2\rangle$. This defines a projection in $\mathcal{A}_\theta$ if and only if f and g satisfy the following relations

$$\begin{aligned} g(e^{2\pi i x^2})g(e^{2\pi i(x^2+\theta)}) &= 0\,, \\ g(e^{2\pi i x^2})[1 - f(e^{2\pi i x^2}) - f(e^{2\pi i(x^2+\theta)})] &= 0\,, \\ f(e^{2\pi i x^2})[1 - f(e^{2\pi i x^2})] &= |g(e^{2\pi i x^2})|^2 + |g(e^{2\pi i(x^2-\theta)})|^2\,. \end{aligned} \tag{A.9}$$

Explicit forms of f, g which satisfy these relations are given as 0follows. Choose any small $\epsilon > 0$ such that $\epsilon < \theta$ and $\theta + \epsilon < 1$, and let $F(x^2) \equiv f(e^{2\pi i x^2})$ for one period be given in the range $x^2 \in [0,1]$ by

$$F(x^2) = \begin{cases} x^2/\epsilon\,, & x^2 \in [0,\epsilon]\,, \\ 1\,, & x^2 \in [\epsilon,\theta]\,, \\ 1-(x^2-\theta)/\epsilon & x^2 \in [\theta,\theta+\epsilon]\,, \\ 0\,, & x^2 \in [\theta+\epsilon,1]\,. \end{cases} \tag{A.10}$$

Then define g for one period by

$$g(e^{2\pi i x^2}) = \begin{cases} \sqrt{F(x^2)(1-F(x^2))}\,, & x^2 \in [0,\epsilon]\,, \\ 0\,, & x^2 \in [\epsilon,1]\,. \end{cases} \tag{A.11}$$

It is easy to see that the functions f and g, defined as the periodic extensions of the above, satisfy the relation (A.9). It can be easily shown that

$$\mathrm{Tr}\ P = \int_0^1 dx^2 \langle x^2|P|x^2\rangle = \int_0^1 dx^2 F(x^2) = \theta\,. \tag{A.12}$$

Thus the projection P now constructed corresponds to P_θ.

Now how about more general projections $P_{n-m\theta}$? Such general projections can be constructed by slightly modifying the above constructed P_θ as follows. The general projection $P_{n-m\theta}$ can be regarded as the projection $P_{\theta'}$ in the algebra $\mathcal{A}_{\theta'}$ if we define $\theta' = n - m\theta$. It is easy to see that the algebra $\mathcal{A}_{\theta'}$ can be embedded in $\mathcal{A}_\theta$ by replacing (U_1, U_2) with (i) (U_1, U_2^m) or (ii) (U_1^m, U_2). Since one can construct the projection $P_{\theta'}$ in the previous way, we obtain the projection $P_{n-m\theta}$ in $\mathcal{A}_\theta$ as desired.

In the first choice (i), the projection is described by functions f and g with period $1/|m|$ and the width of each lump of the function f is given by $(n - m\theta)/|m|$. This preserves the form (A.8) and is called Powers-Rieffel projection [28]. On the other hand, in the second choice (ii) the requirement for being a projection is given by eq. (A.9) with θ replaced by $m\theta$ and this is not included in the form (A.8). Then the width of the lump is enlarged to $n - m\theta$. This construction was given in [14] and used in the proof that any projection can be divided into infinite numbers of mutually orthogonal smaller projections, where the total value of trace is preserved. Note also that in either case, the total area occupied by the lump is $0 \le n - m\theta \le 1$.

Next we turn to the calculation of cyclic 2-cocyle $\tau_2(P)$. Define the derivations δ_1, δ_2 along the two directions of the two-dimensional torus as follows

$$\delta_j U_k = 2\pi i \delta_{jk} U_k\,, \tag{A.13}$$

where δ_{ij} is the ordinary Kronecker's delta. Equivalently, one can express the derivations as follows by using the non-commutative coordinate (x^1, x^2) defined by $U_1 = e^{2\pi i x^1}$, $U_2 = e^{2\pi i x^2}$:

$$\delta_1 = -i\frac{2\pi}{\theta}[x^2,\ ]\,, \qquad \delta_2 = i\frac{2\pi}{\theta}[x^1,\ ]\,. \tag{A.14}$$

From these we can see that δ_1 and δ_2 do commute. Then the cyclic 2-cocyle is defined by

$$\tau_2(a,b,c) = \frac{1}{2\pi i}\,\mathrm{Tr}\left[a\ \delta_1(b)\delta_2(c) - a\ \delta_2(b)\delta_1(c)\right]. \tag{A.15}$$

If we substitute $a = b = c = P_\theta$, then we get [34]

$$\begin{aligned}
\tau_2(P_\theta) &= -(4\pi i)\,\mathrm{Tr}\left[f'(U_2)(g(U_2))^2 - f(U_2)g'(U_2)g(U_2)U_2 + \right.\\
&\qquad\left. + U_1 f(U_2)U_1^* g'(U_2)g(U_2)U_2 - (g(U_2))^2 U_1 f'(U_2)U_2 U_1^*\right]\\
&= -\int_0^1 dx^2 (f(x^2+\theta) - f(x^2))\frac{d}{dx^2}(g(x^2)^2) + \\
&\quad + 2\int_0^1 dx^2 \frac{d}{dx^2}\left(f(x^2+\theta) - f(x^2)\right)\left(g(x^2)^2\right)\\
&= -6\int_0^1 dx^2 \frac{df(x^2)}{dx^2}(g(x^2)^2)\\
&= -6\int_0^1 dt\ (t - t^2) = -1\,.
\end{aligned} \tag{A.16}$$

It is also possible to generalize the result for the projections $P_{n-m\theta}$ and we obtain

$$\tau_2(P_{n-m\theta}) = m\,. \tag{A.17}$$

In order to see this one has only to note that for the description (i) the evaluation of τ_2 is equal to counting of the number of lumps with sign and also that for the description (ii) the factor m is due to the derivation of U_1^m. Indeed this value of τ_2 is the same as that computed from the previously discussed projective modules $E_{n,m}$ $(0 \leq n - m\theta \leq 1)$ which possess the constant curvature $F = \frac{2\pi i m}{n-m\theta}\mathbf{1} \in \mathrm{End}_{\mathcal{A}_\theta} E_{n,m}$ as follows

$$\tau_2(F) = \frac{1}{2\pi i}\,\mathrm{Tr}(F) = m\,. \tag{A.18}$$

Finally let us discuss the relation between the results in the Moyal plane and those in non-commutative torus. Since the radius of the torus is scaled in proportion to $1/\sqrt{\theta}$, the non-commutative torus will approaches the Moyal plane in the limit $\theta \to 0$. In this limit the value of integers n, m which satisfy $0 \leq n - m\theta \leq 1$ is restricted to $n = 0$ or $n = 1$. Thus we obtain the projection $P_{m\theta}$ and $P_{1-m\theta}$. This is consistent with the previous result that in the Moyal plane algebra the projection is given by P_m or $1 - P_m$ up to unitary equivalence.

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142] and references therein.

[3] A. Connes, *Noncommutative geometry*, Academic Press 1994.

[4] R. Minasian and G. Moore, *K-theory and Ramond-Ramond charge*, *J. High Energy Phys.* **11** (1997) 002 [hep-th/9710230].

[5] E. Witten, *D-branes and K-theory*, *J. High Energy Phys.* **12** (1998) 019 [hep-th/9810188].

[6] A review is A. Sen, *Non-BPS states and branes in string theory*, hep-th/9904207.

[7] A. Sen, *Tachyon condensation on the brane antibrane system*, *J. High Energy Phys.* **08** (1998) 012 [hep-th/9805170].

[8] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[9] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as non-commutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031].

JHEP06(2001)041

[10] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[11] Y. Matsuo, *Topological charges of noncommutative soliton*, *Phys. Lett.* **B 499** (2001) 223 [hep-th/0009002].

[12] J.A. Harvey and G. Moore, *Noncommutative tachyons and K-theory*, hep-th/0009030.

[13] E. Witten, *Overview of K-theory applied to strings*, *Int. J. Mod. Phys.* **A 16** (2001) 693 [hep-th/0007175].

[14] I. Bars, H. Kajiura, Y. Matsuo and T. Takayanagi, *Tachyon condensation on noncommutative torus*, *Phys. Rev.* **D 63** (2001) 086001 [hep-th/0010101].

[15] E.M. Sahraoui and E.H. Saidi, *Solitons on compact and noncompact spaces in large noncommutativity*, hep-th/0012259.

[16] E.J. Martinec and G. Moore, *Noncommutative solitons on orbifolds*, hep-th/0101199.

[17] F.P. Boca, *Projections in rotation algebras and theta functions*, *Comm. Math. Phys.* **202** (1999) 325.

[18] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, hep-th/0103256.

[19] Y. Hikida, M. Nozaki and T. Takayanagi, *Tachyon condensation on fuzzy sphere and noncommutative solitons*, *Nucl. Phys.* **B 595** (2001) 319 [hep-th/0008023].

[20] J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060].

[21] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, hep-th/0012145.

[22] C. Kennedy and A. Wilkins, *Ramond-Ramond couplings on brane-antibrane systems*, *Phys. Lett.* **B 464** (1999) 206 [hep-th/9905195].

[23] P. Kraus and F. Larsen, *Boundary string field theory of the* $\mathrm{D}\bar{\mathrm{D}}$ *system*, *Phys. Rev.* **D 63** (2001) 106004 [hep-th/0012198].

[24] T. Takayanagi, S. Terashima and T. Uesugi, *Brane-antibrane action from boundary string field theory*, *J. High Energy Phys.* **03** (2001) 019 [hep-th/0012210].

[25] A. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034].

[26] D. Quillen, *Superconnection and the Chern character*, *Topology* **24** (1985) 89.

[27] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace* C^**-algebra*, Mathematical Surveys and Monographs, 60. Amer. Math. Soc. 1998.

[28] M. Rieffel, *C^*-algebras associated with irrational rotations*, *Pacific J. Math.* **93**(2) (1981) 415.

[29] M. Rieffel, *Projective modules over higher-dimensional non-commutative tori*, *Canad. J. Math.* **40**(2) (1988) 257.

[30] E. Witten, *Noncommutative tachyons and string field theory*, hep-th/0006071.

[31] M. Rieffel and A. Schwarz, *Morita equivalence of multidimensional noncommutative tori*, *Internat. J. Math.* **10** (2) (1999) 289, math.QA/9803057.

[32] B. Pioline and A. Schwarz, *Morita equivalence and T-duality (or B versus Θ)*, *J. High Energy Phys.* **08** (1999) 021 [hep-th/9908019].

[33] M.V. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain crossed-product C^* algebras*, *J. Operator Theory* **4** (1980) 93; *Imbedding the irrational rotation C^*-algebra into an AF-algebra*, *J. Operator Theory* **4** (1980) 201.

[34] A. Connes, *C^* algebras and differential geometry*, *Compt. Rend. Acad. Sci. (Ser. I Math.)* **A290** (1980) 599 [hep-th/0101093].

[35] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[36] P.-M. Ho, *Twisted bundle on quantum torus and BPS states in matrix theory*, *Phys. Lett.* **B 434** (1998) 41 [hep-th/9803166];
B. Morariu and B. Zumino, *Super Yang-Mills on the noncommutative torus*, hep-th/9807198;
D. Brace, B. Morariu and B. Zumino, *Dualities of the matrix model from T-duality of the type-II string*, *Nucl. Phys.* **B 545** (1999) 192 [hep-th/9810099].

[37] C. Hofman and E. Verlinde, *U-duality of Born-Infeld on the noncommutative two-torus*, *J. High Energy Phys.* **12** (1998) 010 [hep-th/9810116].

[38] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rep.* **244** (1994) 77 [hep-th/9401139].

[39] A. Sen, *Supersymmetric world-volume action for non-BPS D-branes*, *J. High Energy Phys.* **10** (1999) 008 [hep-th/9909062].

[40] A. Sen, *BPS D-branes on non-supersymmetric cycles*, *J. High Energy Phys.* **12** (1998) 021 [hep-th/9812031].

[41] A. Connes and M. Rieffel, *Yang-Mills for non-commutative two-tori*, *Contemp. Math.* **62**1987237.

[42] E. Witten, *On background independent open string field theory*, *Phys. Rev.* **D 46** (1992) 5467 [hep-th/9208027]; *Some computations in background independent off-shell string theory*, *Phys. Rev.* **D 47** (1993) 3405 [hep-th/9210065].

JHEP06(2001)041

[43] A.A. Gerasimov and S.L. Shatashvili, *On exact tachyon potential in open string field theory*, *J. High Energy Phys.* **10** (2000) 034 [hep-th/0009103].

[44] D. Kutasov, M. Marino and G. Moore, *Some exact results on tachyon condensation in string field theory*, *J. High Energy Phys.* **10** (2000) 045 [hep-th/0009148]; *Remarks on tachyon condensation in superstring field theory*, hep-th/0010108.

[45] K. Ichikawa and Y. Matsuo work in progress.

[46] S. Mukhi and N.V. Suryanarayana, *Gauge-invariant couplings of noncommutative branes to Ramond-Ramond backgrounds*, *J. High Energy Phys.* **05** (2001) 023 [hep-th/0104045]; *Chern-Simons terms on noncommutative branes*, *J. High Energy Phys.* **11** (2000) 006 [hep-th/0009101].

[47] T. Krajewski and M. Schnabl, *Exact solitons on noncommutative tori*, hep-th/0104090.